Time-ordered fluctuation-dissipation relation for incompressible isotropic turbulence

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The Kraichnan-Wyld perturbation expansion is used to justify the introduction of a renormalized response function connecting two-point covariances at different times. The resulting relationship was specialized by a suitable choice of initial conditions to the form of a fluctuation-dissipation relation (FDR). This was further developed to reconcile the time symmetry of the covariance with the causality of the response by the introduction of time ordering along with a counterterm. This formulation provides a solution to an old problem, that of representing the time dependence of the covariance and response by exponential forms. We show that the derivative (with respect to difference time) of the covariance now vanishes at the origin. This allows one to study the relationships between two-time spectral closures and time-independent theories such as the self-consistent field theory of Edwards or the more recent renormalization group approaches. We also show that the renormalized response function is transitive with respect to intermediate times and report a different Langevin equation model for turbulence. We note the potential value of this time-ordering procedure in all applications of the FDR.

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Most renormalized spectral theories of turbulence have been based on the Kraichnan-Wyld perturbation theory [1,2]. Let us denote the two-time covariance of the fluctuating turbulent velocity field for mode k by $C(k;t,t')$, where the dependence is only on the scalar wave number due to the assumption of isotropy. The covariance is related to the usual energy spectrum $E(k,t)$ by

$$E(k,t) = 4\pi k^2 C(k;t,t). \tag{1}$$

Isotropy also implies time-reversal symmetry, which requires that

$$C(k;t,t') = C(k;t',t). \tag{2}$$

Reversion of the primitive perturbation series, obtained by iterating the Navier-Stokes equation (NSE) for the bare covariance $C^{(0)}(k;t,t')$ (which has a multivariate normal distribution and is not an observable) in terms of the viscous response (which is an observable), leads to coupled expansions for the exact covariance $C(k;t,t')$ and a renormalized response function $R(k;t,t')$ (say). The renormalized response is not an observable but must nevertheless satisfy the causality condition

$$R(k;t,t') = 0 \text{ for } t' > t. \tag{3}$$

Specific theories are obtained by introducing a specific choice of $R(k;t,t')$ and truncating the renormalized expansions at some low order. The first such theory was the Eulerian [20] direct interaction approximation (DIA: Ref. [1]), in which the response to small perturbations in the forcing (noise) is renormalized. Other pioneering theories are the self-consistent field theories [3,4]. These theories are time independent [21] and the renormalized response is expressed in terms of the eddy decay rate $\omega(k)$. It was shown [5] (see also Ref. [6] for a discussion) that a connection could be made between these approaches by considering the steady state [where $C(k;t,t')=C(k;t-t')$], and assuming exponential forms for the covariance and renormalized response function, thus

$$C(k;t-t') = C(k)e^{-\omega(k)|t-t'|}; \tag{4}$$

$$R(k;t-t') = e^{-\omega(k)(t-t')}. \tag{5}$$

However, there is a basic problem with these forms in that the time-reversal symmetry of Eq. (2) is in practice not satisfied, and that differentiating the steady-state covariance with respect to difference time leads to a nonzero result at the origin, where $t=t'$ (see Ref. [6]). In this paper we will show that a consideration of time ordering in the renormalized response can allow the use of exponential time dependences without encountering these problems.

We begin by considering the generalized covariance equation, as derived from the NSE [7], thus

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] C_{\alpha\sigma}(\mathbf{k};t,t') = \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \times \int d^3 j \langle u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) u_\sigma(-\mathbf{k},t')\rangle. \tag{6}$$

The Greek indices are just the usual Cartesian tensor indices relating to the space dimensions and take the values 1, 2, or 3. The inertial transfer operator $M_{\alpha\beta\gamma}(\mathbf{k})$ (see, for example, Ref. [7]) is given by

$$M_{\alpha\beta\gamma}(\mathbf{k}) = \frac{1}{2i}[k_\beta P_{\alpha\gamma} + k_\gamma P_{\alpha\beta}], \tag{7}$$

where the projector $P_{\alpha\beta}(\mathbf{k})$ is given by

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$$P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}. \tag{8}$$

By using an integrating factor and integrating over time we can write this as

$$C_{\alpha\sigma}(\mathbf{k};t,t') = R^{(0)}_{\alpha\varepsilon}(\mathbf{k};t,s)C_{\epsilon\sigma}(\mathbf{k};s,t') + \left[\lambda\int_s^t dt'' R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,t'')M_{\epsilon\beta\gamma}(\mathbf{k}) \times \int d^3j\langle u_\beta(\mathbf{j},t'')u_\gamma(\mathbf{k}-\mathbf{j},t'')u_\sigma(-\mathbf{k},t')\rangle\right], \tag{9}$$

where s is some initial time and the integrating factor is

$$R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,t'') = \begin{cases} P_{\alpha\epsilon}(\mathbf{k})e^{-\nu k^2(t-t'')} & t \geqslant t'', \\ 0 & t < t''. \end{cases} \tag{10}$$

From the primitive perturbation series [1,2], we have

$$C_{\alpha\sigma}(\mathbf{k};t,t') = C^{(0)}_{\alpha\sigma}(\mathbf{k};t,t') + \lambda^2 C^{(2)}_{\alpha\sigma}(\mathbf{k};t,t')\cdots. \tag{11}$$

When Eq. (11) is substituted in Eq. (9) we can see that $R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,s)$ acts as a zero-order response for the zero-order covariance, thus

$$C^{(0)}_{\alpha\sigma}(\mathbf{k};t,t') = \theta(t-s)R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,s)C^{(0)}_{\epsilon\sigma}(\mathbf{k};s,t'). \tag{12}$$

This is an exact result. We shall call this the zero-order or bare result. Re-arranging Eq. (9) to prompt the next step,

$$C_{\alpha\sigma}(\mathbf{k};t,t') = \Bigg[R^{(0)}_{\alpha\varepsilon}(\mathbf{k};t,s) + \frac{1}{C_{\epsilon\sigma}(\mathbf{k};s,t')}\lambda\int_s^t dt'' R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,t'')M_{\epsilon\beta\gamma}(\mathbf{k}) \times \int d^3j\langle u_\beta(\mathbf{j},t'')u_\gamma(\mathbf{k}-\mathbf{j},t'')u_\sigma(-\mathbf{k},t')\rangle\Bigg] \times C_{\epsilon\sigma}(\mathbf{k};s,t'), \tag{13}$$

we postulate that we may write this in its renormalized form as

$$C_{\alpha\sigma}(\mathbf{k};t,t') = \theta(t-s)R_{\alpha\epsilon}(\mathbf{k};t,s)C_{\epsilon\sigma}(\mathbf{k};s,t'), \tag{14}$$

or in its *isotropic* version as

$$C(k;t,t') = \theta(t-s)R(k;t,s)C(k;s,t'), \tag{15}$$

where the $\theta(t-s)$ incorporates the causality condition. We have effectively replaced the zero-order equation (12) by its renormalized version using the replacements

$$C^{(0)} \to C, \qquad R^{(0)} \to R. \tag{16}$$

As yet we have made no choice about the time ordering of the two times t and t', and thus the symmetry under interchange of t and t' is untested in Eq. (15). If we explicitly choose the time ordering as $t > t'$ say, then this is equivalent to applying the Heaviside unit-step function $\theta(t-t')$ to both sides of Eq. (15)

$$\theta(t-t')C(k;t,t') = \theta(t-t')\theta(t-s)R(k;t,s)C(k;s,t'). \tag{17}$$

If we now set $s = t'$ in Eq. (17), which amounts to a choice of the initial condition, we get

$$\theta(t-t')C(k;t,t') = \theta(t-t')R(k;t,t')C(k;t',t'). \tag{18}$$

This result takes the form of a fluctuation-dissipation relationship (or FDR). Of course such relationships are most familiar in microscopic systems at thermal equilibrium but over the years there has been quite some discussion of the way in which relationships like this occur in turbulence theory (for example, see Ref. [8] and references therein). More recently Frederiksen and Davies [9] have distinguished between spectral theories by the way in which relationships of the form of Eq. (18) play a part.

We now introduce a representation of the covariance which preserves the symmetry under interchange of time arguments as

$$C(k;t,t') = \theta(t-t')C(k;t,t') + \theta(t'-t)C(k;t,t') - \delta_{t,t'}C(k;t,t'). \tag{19}$$

Using Eq. (17) to expand the right-hand side of Eq. (19) we obtain

$$C(k;t,t') = \theta(t-t')\theta(t-s)R(k;t,s)C(k;s,t') + \theta(t'-t)\theta(t'-p)R(k;t',p)C(k;p,t) - \delta_{t,t'}C(k;t,t'). \tag{20}$$

Equation (20) may be written in the form of a time-ordered fluctuation-dissipation relation by using Eq. (18) to construct it instead,

$$C(k;t,t') = \theta(t-t')R(k;t,t')C(k;t',t') + \theta(t'-t)R(k;t',t)C(k;t,t) - \delta_{t,t'}C(k;t,t'). \tag{21}$$

The symmetry of both these covariances, Eqs. (20) and (21), can be broken simply by applying a unit-step function to both sides. This will yield something like Eq. (17) or (18), respectively, depending on which time ordering is chosen.

Turning now to the problem of the exponential forms as given by Eqs. (4) and (5) we find that this time-ordered representation (19) has the property that

$$\lim_{t\to t'}\frac{\partial}{\partial t}C(k;t,t') = 0, \tag{22}$$

as it should for stationary turbulence. We can show this result by using Eq. (5) in Eq. (21) to obtain

$$C(k;t,t') = \theta(t-t')\exp\{-\omega(k)(t-t')\}C(k;t',t') + \theta(t'-t)\exp\{-\omega(k)(t'-t)\}C(k;t,t) - \delta_{t,t'}C(k;t,t'). \tag{23}$$

Using the property of stationary turbulence

$$C(k;t,t) = C(k), \tag{24}$$

and the representation for $\delta_{t,t'}$

$$\delta_{t,t'} = \theta(t-t')\theta(t'-t), \tag{25}$$

Eq. (23) becomes

$$\begin{aligned} C(k;t,t') = \theta(t-t')&\exp\{-\omega(k)(t-t')\}C(k) \\ &+ \theta(t'-t)\exp\{-\omega(k)(t'-t)\}C(k) \\ &- \theta(t-t')\theta(t'-t)C(k;t,t'). \end{aligned} \tag{26}$$

Taking the derivative with respect to t, substituting Eq. (21) in places, collecting all the terms and taking the limit $t\to t'$, we find

$$\begin{aligned} \lim_{t\to t'}\frac{\partial}{\partial t}C(k;t,t') = &-C(k)\omega(k) + C(k)\lim_{t\to t'}\delta(t-t') \\ &+ C(k)\omega(k) - C(k)\lim_{t\to t'}\delta(t'-t) \\ &+ C(k)\omega(k) - C(k)\lim_{t\to t'}\delta(t'-t) \\ &- C(k)\omega(k) + C(k)\lim_{t\to t'}\delta(t-t'). \end{aligned} \tag{27}$$

Using the fact that the δ function behaves like an even function, namely $\delta(t-t') = \delta(t'-t)$, we can see that the terms cancel in pairs and we are left with the desired result Eq. (22).

In contrast to this, the use of Eq. (4) by Leslie failed to obtain this result [6]. This is because the representation in Eq. (21) exhibits the time-reversal symmetry $t\leftrightarrow t'$ in a more manifest way than Eq. (4). Leslie, for his calculation regarding the time derivative of the stationary covariance at the origin, takes $t>t'$ for the representation of the covariance, whereas we have determined in Eq. (2) that the covariance is symmetric under interchange of t and t'.

Finally, we can show that the renormalized response is transitive with respect to intermediate times. Equating the right-hand side of Eq. (17) with the right-hand side of Eq. (18) we obtain

$$\begin{aligned} \theta(t-t')R(k;t,t')C(k;t',t') = \theta(t-t')\theta(t-s) \\ \times R(k;t,s)C(k;s,t'). \end{aligned} \tag{28}$$

Expanding the right-hand side of Eq. (28) using Eq. (21) we have

$$\begin{aligned} &\theta(t-t')R(k;t,t')C(k;t',t') \\ &\quad = \{[\theta(t-t')\theta(t-s)R(k;t,s) \\ &\qquad \times\theta(s-t')R(k;s,t')C(k;t',t')]\}_a + \{[\theta(t-t')\theta(t-s) \\ &\qquad \times R(k;t,s)\theta(t'-s)R(k;t',s)C(k;s,s)]\}_b \\ &\qquad - \{[\theta(t-t')\theta(t-s)R(k;t,s)\delta_{t',s}C(k;s,t')]\}_c. \end{aligned} \tag{29}$$

Specializing to the case $t>s>t'$ corresponds to $\{\cdot\}_b=0$ and $\{\cdot\}_c=0$, leaving

$$\begin{aligned} &\theta(t-t')R(k;t,t')C(k;t',t') \\ &\quad = [\theta(t-t')\theta(t-s)R(k;t,s)\theta(s-t')R(k;s,t')C(k;t',t')]. \end{aligned} \tag{30}$$

We now use the contraction property of the Heaviside function $\theta(t-s)\theta(s-t') = \theta(t-t')$ to write Eq. (30) as

$$\begin{aligned} \theta(t-t')&\underline{R(k;t,t')}C(k;t',t') \\ &= \theta(t-t')\underline{R(k;t,s)R(k;s,t')} \\ &\quad \times C(k;t',t'). \end{aligned} \tag{31}$$

[The underlined areas denote the origin of Eq. (32).] From this above result, we can deduce the transitive property of the renormalized response

$$R(k;t,t') = R(k;t,s)R(k;s,t'). \tag{32}$$

These may seem like small results but the fact is that the renormalized perturbation theories of turbulence, which looked so promising initially, have been essentially in a static state for at least three decades. Sometimes the subject is described as being "mired in controversy" but in reality a sober appraisal reveals only a few minor unresolved issues. With the difficulties of using exponential representations of time dependences resolved, the way is clear to explore and learn from the relationships between the different classes of theories. This also includes renormalization group methods, where there is a relationship between DIA [1] and the method of iterative averaging [10,11].

In this short communication we have already shown that our time-ordering approach can be used to prove the transitivity of the renormalized response function. This is not, in itself, a trivial result and moreover has important implications for the dimensional reduction of this type of theory [12]. We have also used these methods to derive a different Langevin equation model of turbulence. With the *ansatz* of local energy transfer to determine the response (see Ref. [13] and references therein), along with an assumption of an exponential relationship between the response function and the eddy damping, as given by Eq. (5), we find

$$\begin{aligned} \left(\frac{\partial}{\partial t} + 2\nu k^2\right)C(k;t) &= 2\int d^3j L(\mathbf{k},\mathbf{j})D(k,j,|\mathbf{k}-\mathbf{j}|;t)C(|\mathbf{k}-\mathbf{j}|;t) \\ &\quad \times[C(j;t) - C(k;t)] \\ &= -2\omega(k;t)C(k;t), \end{aligned} \tag{33}$$

$$\begin{aligned} \omega(k;t) = -\int d^3j L(\mathbf{k},\mathbf{j})D(k,j,|\mathbf{k}-\mathbf{j}|;t)\,\frac{C(|\mathbf{k}-\mathbf{j}|;t)}{C(k;t)} \\ \times[C(j;t) - C(k;t)], \end{aligned} \tag{34}$$

and

$$\frac{\partial D(k,j,|\mathbf{k}-\mathbf{j}|;t)}{\partial t} = 1 - [(\nu k^2 + \nu j^2 + \nu|\mathbf{k}-\mathbf{j}|^2) + \omega(k;t) + \omega(j;t) + \omega(|\mathbf{k}-\mathbf{j}|;t)]D(k,j,|\mathbf{k}-\mathbf{j}|;t), \tag{35}$$

where

$$L(\mathbf{k},\mathbf{j}) = -2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\alpha\delta}(\mathbf{j})P_{\gamma\delta}(\mathbf{k}-\mathbf{j}). \tag{36}$$

The initial conditions can be taken as

$$C(k;t=0) = \frac{E(k;t=0)}{4\pi k^2}, \tag{37}$$

where $E(k;t=0)$ is some arbitrarily chosen initial energy spectrum, and

$$D(k,j,|\mathbf{k}-\mathbf{j}|;t=0) = 0. \tag{38}$$

This is similar to the test-field model [14], but has an extra term in the equation for the eddy damping. The extra term cancels infrared divergences and this means that (unlike the test-field model) it does not require an additional hypothesis and adjustable constant to be compatible with the Kolmogorov distribution. An account of this work is in preparation.

Finally, in the interests of completeness, we should mention that following the seminal paper of Leith [8] the fundamental issues involved in obtaining fluctuation-dissipation relationships for chaotic systems have received attention, particularly from the point of view of dynamical systems theory [15–19]. It has been shown that a general fluctuation-dissipation relationship exists for systems which are mixing. The precise form of the relationship for any specific system is found to depend on the invariant measure of that system. The relationship of our present work to this existing activity raises several interesting questions and we intend to address these in our forthcoming paper mentioned above.

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